Introduction

Introduction

- Definition.
  - **Paradox:** *A paradox is a true statement (or group of statements) that leads to a contradiction or a situation which defies intuition; or, inversely, it can be an apparent contradiction that actually expresses a truth.*

    The word 'paradox' is often used interchangeably with contradiction.

    Often, mistakenly, it is used to describe situations that are ironic.
  - **Antinomy:** Statement that reaches a self-contradictory result by properly applying accepted ways of reasoning.
  - **Dialetheia:** A paradox which is both true and false at the same time.
  - Some have led to advances in science, philosophy and mathematics; others still remain unresolved.
  - Common themes in paradoxes include (amongst many others):
    - self-reference,
    - the infinite,
    - circular definitions,

Zeno's Paradox

Zeno's paradox

- **Proposition:** Suppose I am a teacher that wants to examine my pupils with a surprise exam some time this week, which day would be the least expected for it to take place?
  - **SOLUTION:** We have to discard Friday, because when Thursday arrives and no exam has taken place, it will be obvious that the chosen day was Friday. In that case, Thursday comes in to be effectively the new last day of the week. So now, on Wednesday, my pupils will know for sure the exam is on Thursday and there'll be no surprise again. So Thursday has to be discarded, and so on. This reasoning will give us no possible day at all in the end.
  - Looking back to our definition of paradox: This solutions consists of a series of true statements that in the end, defy intuition
  - This is part of a number of "infinite division" paradoxes..

Zeno's paradox

- Achilles and the tortoise:
  - *"In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."* —Aristotle, Physics VI:9, 239b15

  - In the paradox of Achilles and the Tortoise, Achilles is in a footrace with the tortoise. Achilles allows the tortoise a head start of 100 feet. If we suppose that each racer starts running at some constant speed (one very fast and one very slow), then after some finite time, Achilles will have run 100 feet, bringing him to the tortoise's starting point. During this time, the tortoise has run a much shorter distance, for example 10 feet. It will then take Achilles some further time to run that distance, in which time the tortoise will have advanced further; and then more time still to reach this third point, while the tortoise moves ahead. Thus, whenever Achilles reaches somewhere the tortoise has been, he still has further to go. Therefore, because there are an infinite number of points Achilles must reach where the tortoise has already been--he can never overtake the tortoise.

Zeno's paradox

- A different aproach to the same problem:
  - Supposition:
    - You want to cross a room. First, you would have to cover half the distance from one side to the other. In order to cover this new distance it is obvious that you would have to walk half of it first, which is a quarter of the total distance. And to cover this new distance, you would have to walk and eigth of the total, a sixteenth... and so on forever.
  - Consequence:
    - You would never be able to reach the other side. **Motion** has become impossible because moving involves to cover an infinite number of very small distances.
    - ***In this case, the tortoise would won the race.***

Zeno's paradox

- Resolution:
  - As the movement is, in reality, possible, a doubt arises:
    - What's wrong with Zeno? Where's the error in his logic?
  - "Adding up an infinite number of positive numbers results in infinite"

    --> not always true
  - Suppose the total distance is given a value, for example 2km. Then, we would have to walk 1km first, then 1/2, then 1/4.... and so on.

If one walks all those distances, what total distance would have walked?

SUM $\left\{\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right\}$ =2

- Now, as there is indeed a finite total time to cover the total distance... and considering Achilles is faster, **he would have won de race!**

Hotel Paradox

Hotel Paradox

- David Hilbert (based on Cantor's transinfinite number theory)

- **PROBLEM:**
  - 2 great hostesses that wanted to build the biggest hotel in the world, met to talk about how many rooms it should have.

    "What about 1000 rooms?"

    "No, because if someone builts one with 2000 rooms, ours would be small. Let's make it 10.000."

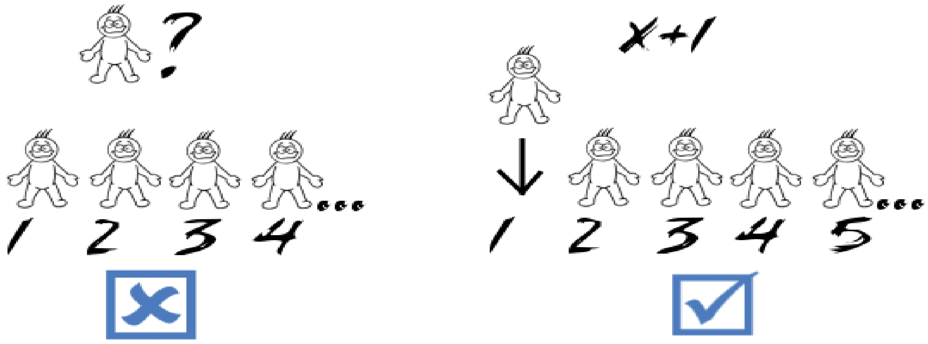
    "But it could happen that someone builts one with 20.000 so then, we sould still have a small hotel. Let0s make it 1.000.000 rooms!"

    "And what if someone...... "

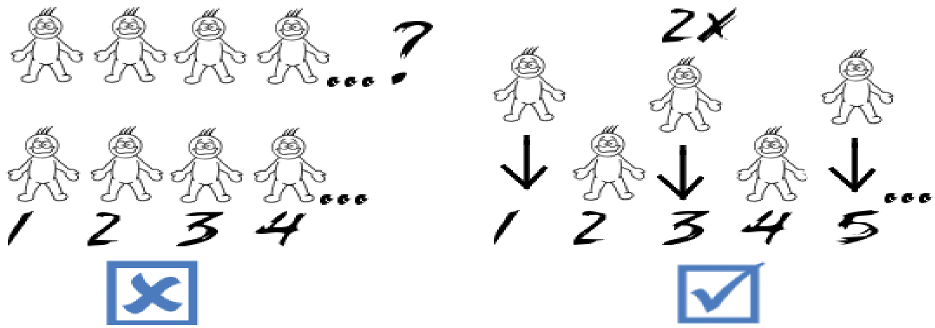
  - As there is always a possibility of building something bigger, they concluded that it was necessary to build a hotel with infinite rooms.

Hotel Paradox

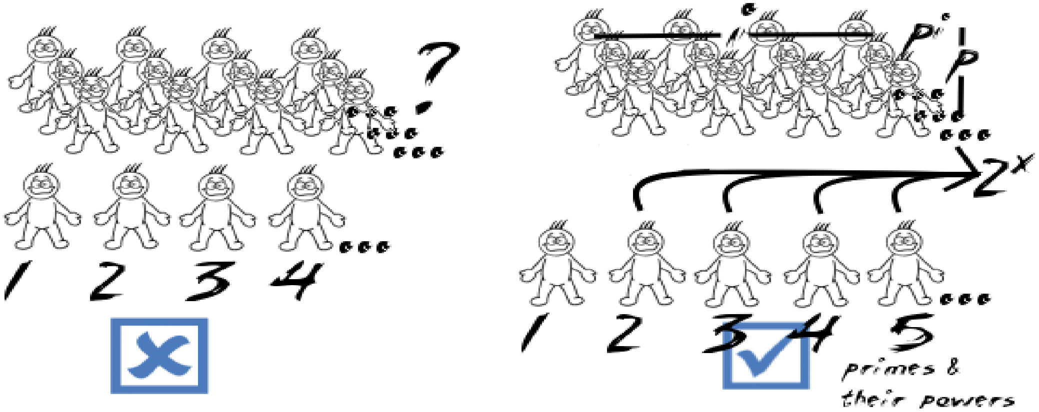
- Paradox cases:
  - **Infinite + 1:**



  - **Double infinites:**



  - **Infinite infinites:**



Exercise

Thompson's Lamp

- **PROPOSITION:**
  - Consider a Lamp with a switch. You clic on the switch once to light the lamp, and if you clic again, it is switched off… simple.

  Imagine you like to play with the lamp this way:

  First, switch it on. 1 minute later, switch it off. Half a minute later switch it on again. 15 seconds later, switch it off again… and so on.

  It's easy to see with the theory explained that it will all add up to 2 minutes.

  - **So, after 2 minutes, will the lamp be ON or OFF?**

  - Would it make any difference if it was ON for a start?

Thompson's Lamp

- **SOLUTION:**

(a) The bulb would burn out.

(b) The bulb will *seem* lighted, because of the persistence of light in the eye.

(c) Switching faster than a 0.02 second period (1/50Hz corresponding to the electrical power frecuency signal) the bulb would light up with less intensity as speed increases, because there would be no time for it to get brighter, so it will tend to not lighting up at all. But at the same time the heat remaining on the filament will keep it lighted until the next time it is switched on so it will not really quit being lighted, This is a paradox itself.

(d) When arriving to the speed of light in switching ON and OFF, we would turn back instances of time up to the point of switching ON the bulb while it was already lighted. It will become an anticausal system.

(e) We can't tell.

Russel's Paradox

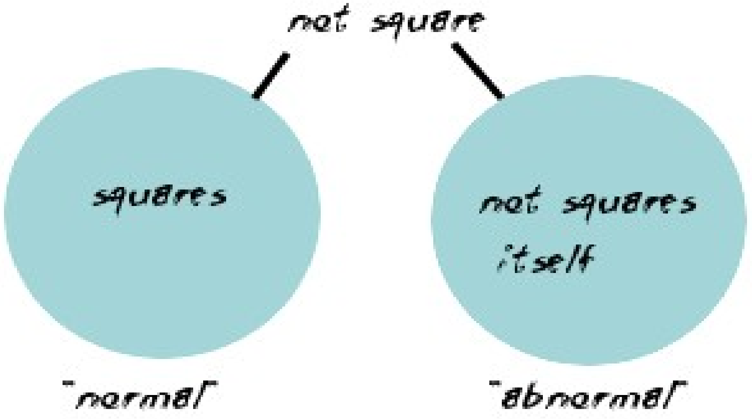
Russel's Paradox

- The set of all sets which are not elements of themselves. (which includes, and therefore does not, and therefore does include itself).
  - CONTRADICTION: On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition.

Let us call a set "abnormal" if it is a member of itself, and "normal" otherwise. For example, take the set of all squares.

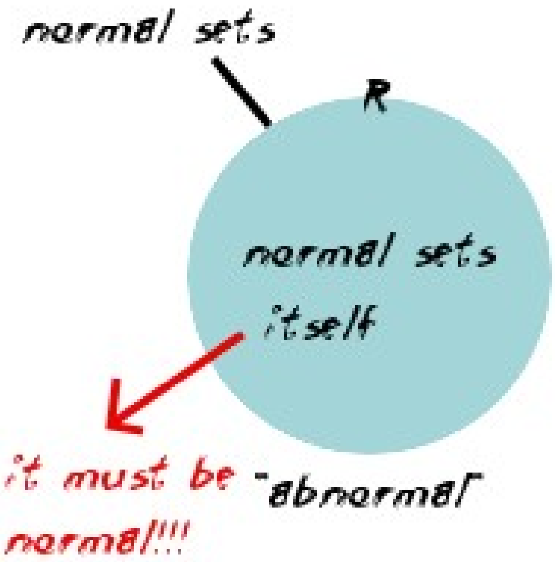
That set is not itself a square, and therefore is not a member of the set of all squares. So it is "normal".

On the other hand, if we take the complementary set of all non-squares, that set is itself not a square and so should be one of its own members. It is "abnormal



Russel's Paradox

- Now we consider the set of all normal sets – let us give it a name: R. If R were abnormal, that is, if R were a member of itself, then since R only contains normal sets, R must be normal, which is contradictory to our original hypothesis: R is abnormal. So, R cannot be abnormal, which means R is normal. Paradoxically, we are led to the contradiction that R is both normal and abnormal.



Barber Paradox

- A different approach of Russel's Paradox:
  - Suppose there is a town with just one male barber; and that every man in the town keeps himself clean-shaven: some by shaving themselves, some by attending the barber. It seems reasonable to imagine that the barber obeys the following rule: He shaves only those men who do not shave themselves.
  - Under this scenario, we can ask the following question: Does the barber shave himself?

"In my town I'm the only barber. If I shave myself, then I can shave myself, so the barber in my town shouldn't shave me... but that's me! But, on the contrary, if I don't shave myself, then the barber should. ¡But I'm the only barber in my town!"

  - The situation presented is in fact impossible:

Beatriz Cabrera

beacabfi@teleco.upv.es